International Journal of Theoretical Physics, Vol. 9, No. 3 (1974), pp. 157–165

The Contribution of the *C*-Field to the Action Functional

J. M. HOBBS

Department of Mathematics and Computer Studies, Sunderland Polytechnic

Received: 27 *February* 1973

Abstract

It is shown that the action of the usual field theory requires the inclusion of two terms in order to be equivalent, in the macroscopic case, to the action proposed by Hoyle & Narlikar (1964c). These actions correspond only to a modified form of Maxwell's equations, which, in consequence, lose their property of conformal invariance. It is also demonstrated how the *C*-field and electromagnetic field can be brought into unison by an appropriate re-definition of the vector potential. Both field theories can thus be described in terms of one vector Green's function.

1. *Introduction*

In two recent papers Hoyle & Narlikar (1964a, b) have described a method of overcoming singularities in the gravitational equations. The first paper (1964a) showed that the presence of a *C*-field in these equations, satisfying the source equation, actually prevents singularities from occurring. Whilst, in their second paper (1964b), they proceeded to construct a *C*-field, defined in terms of a scalar Green's function, which satisfied the source equation identically and was therefore of the type required to overcome the singularity problem. In a later paper (1964c) the above authors found that, by defining the *C*-field and the vector potential in terms of specific Green's functions, they were able to simplify the usual form of action functional. This was due to the fact that, by choosing the Green's functions correctly, they could reduce both Maxwell's equations and the source equation to identities. This means that the only equations which actually emerge from their action functional are those for the gravitational field. These were found to contain additional terms which could not be evolved from the action of the usual field theory.

In the work that follows we shall show that these additional terms arise because the two actions we are considering are mutually incompatible. In fact, in order to obtain agreement with the simplified action it is necessary to include two extra terms into the action of the usual field theory. These are found to account for the additional terms appearing in the energy-momentum tensor of Hoyle and Narlikar. Two very important points which emerge from this work are that the two actions correspond only to a modified form of Maxwell's equations and that the modified Maxwellian equations are no longer conformally invariant. This additional contribution of the creation field to the current density does mean, however, that the C-field can be absorbed into electromagnetic theory by an appropriate re-definition of the vector potential. In consequence, only one Green's function is required in the action in order to describe both the electromagnetic field and the C-field.

2. *The Two Actions*

Throughout this work we shall comply with the notation and terminology of Hoyle & Narlikar (1964c). The world lines of particles will be labelled a, b, etc., and a typical point on world line 'a' will be denoted by A. The indices of the various tensors associated with A will carry a suffix A, for example i_A, k_A. The coordinates of A are a^{i_A} and the proper time at A will be denoted by 'a' with

$$da^2 = g_{i_A k_A} da^{i_A} da^{k_A} \tag{2.1}$$

Now consider the world line 'a' to be a segment with ends at the points A_1, A_2 and with A_2 at the later time. We shall define the contribution of the world line 'a' to the total creation field by

$$\begin{aligned} C^{(a)}(X) &= f^{-1}\{\bar{G}(X, A_2) - \bar{G}(X, A_1)\} \\ &= f^{-1} \int_{A_1}^{A_2} \bar{G}_{.i_A} \dot{a}^{i_A} da \end{aligned} \tag{2.2}$$

where $\bar{G}$ is the scalar Green's function which is a solution of the wave equation

$$g^{i_A k_A} \bar{G}(A, B)_{.i_A k_A} = -\bar{g}^{-1/2} \delta^{(4)}(A, B) \tag{2.3}$$

and f is a coupling constant. Dots over the coordinates denote absolute covariant differentiation with respect to the proper time, $\delta^{(4)}(A, B)$ is the four-dimensional Dirac delta function and $\bar{g}$ denotes minus the determinant of the parallel propagator. The total creation field at X due to all world lines is

$$C(X) = \sum_a C^{(a)}(X) \tag{2.4}$$

Our next problem is to introduce the vector potential at the point X due to a typical world line 'a'. This is achieved by forming Maxwell's equations from the action of the usual field theory, that is from

$$L = \frac{1}{16\pi G}\int R\sqrt{(-g)}\,d^4x - \sum_a m_a \int da$$
$$- \frac{1}{16\pi}\int F_{lm}F^{lm}\sqrt{(-g)}\,d^4x - \sum_a e_a \int A_i\,da^i$$
$$- \tfrac{1}{2}f\int C_{.i}C^{.i}\sqrt{(-g)}\,d^4x + \sum_a \int C_{.i}\,da^i \tag{2.5}$$

where A_i is the electromagnetic 4-potential. An independent variation of the A_i, keeping the geometry and C fixed, leads to Maxwell's equations in the form

$$F^{ij}{}_{.j} = 4\pi j^i \tag{2.6}$$

where the field strengths are

$$F_{ij} = A_{j.i} - A_{i.j} \tag{2.7}$$

and j^i is the charge-current four vector given by

$$j^i = \sum_a e_a \int \bar{g}^{-1/2}\delta^{(4)}(A, X)\,\bar{g}^{i_x}{}_{i_A}\,\dot{a}^{i_A}\,da \tag{2.8}$$

On the other hand the action used by Hoyle & Narlikar (1964c) takes the simplified forms

$$A = \frac{1}{16\pi G}\int R\sqrt{(-g)}\,d^4x - \sum_a m_a \int da$$
$$- \tfrac{1}{2}\sum_a e_a \int \sum_{b\neq a} A^{(b)}{}_i\,da^i + \tfrac{1}{2}\sum_a \int \sum_{b\neq a} C^{(b)}{}_{.i}\,da^i \tag{2.9}$$
$$= \frac{1}{16\pi G}\int R\sqrt{(-g)}\,d^4x - \sum_a m_a \int da$$
$$- \sum_{a<b}\sum 4\pi e_a e_b \iint \bar{G}_{i_A i_B}\,da^{i_A}\,db^{i_B}$$
$$+ f^{-1}\sum_{a<b}\sum \iint \bar{G}_{.i_A i_B}\,da^{i_A}\,db^{i_B} \tag{2.10}$$

where $\bar{G}_{i_A i_B}$ is a vector Green's function which we shall define later using Maxwell's equations. In (2.9) and (2.10) the fields are not given independent degrees of freedom, since they re interrelated through the geometry of the system. This means that only the gravitational equations can emerge from the action functional. However, Hoyle and Narlikar assert that these actions still remain equivalent to the action (2.5), of the usual field theory, by virtue of the fact that, due to the choice of vector and scalar Green's functions, both Maxwell's equations and the source equation for the current density are satisfied identically. Let us now investigate this statement to see whether this can indeed be the case.

Substituting equation (2.7) into equation (2.6) we see that Maxwell's equations for the usual field theory become

$$g^{j_x k_x} A^{(a)}{}_{i_x \cdot j_x k_x} - g_{i_x}{}^{k_x} A^{(a)}{}_{j_x \cdot}{}^{j_x}{}_{k_x} + R_{i_x}{}^{k_x} A^{(a)}{}_{k_x} = -4\pi j^{(a)}{}_{i_x} \tag{2.11}$$

where $A^{(a)}{}_{i_x}$ and $j^{(a)}{}_{i_x}$ represent the contributions of the world line 'a' to the total vector potential and current four vector respectively. In solving equation (2.11) we shall not restrict ourselves to a particular gauge, but instead define an arbitrary gauge by

$$A^{(a)k_x}{}_{.k_x} = 4\pi e_a \int \Lambda_{.i_A} \dot{a}^{i_A}\, da \tag{2.12}$$

where $\Lambda(X, A)$ is an arbitrary bi-scalar.

We next define a solution of equation (2.11) for the vector potential in terms of the vector Green's function by the relation

$$A^{(a)}{}_{k_x} = 4\pi e_a \int \bar{G}_{k_x i_A} \dot{a}^{i_A}\, da \tag{2.13}$$

and thus assert that the vector Green's function satisfies the inhomogeneous wave equation

$$g^{j_x k_x} \bar{G}_{i_x i_A \cdot j_x k_x} - \Lambda_{.i_x i_A} + R_{i_x}{}^{k_x} \bar{G}_{k_x i_A} = -\bar{g}^{-1/2} \bar{g}_{i_x i_A} \delta^{(4)}(X, A) \tag{2.14}$$

There exists an extremely important relation between the scalar and vector Green's functions which can be obtained by using the identity

$$(\bar{g}^{-1/2} \bar{g}^{i_x}{}_{i_A} \delta^{(4)}(X, A))_{.i_x} = -(\bar{g}^{-1/2} \delta^{(4)}(X, A))_{.i_A} \tag{2.15}$$

and taking the divergence of (2.14). Taking equation (2.3) into account, we find

$$\begin{aligned} -g^{i_x j_x} \bar{G}_{.i_A i_x j_x} &= g^{i_x j_x} \bar{G}^{k_x}{}_{i_A \cdot i_x j_x k_x} \\ &\quad - \Lambda_{.}{}^{i_x}{}_{i_A i_x} + (R^{k_x}{}_{j_x} \bar{G}^{j_x}{}_{i_A})_{.k_x} \\ &= g^{i_x j_x} (\bar{G}^{k_x}{}_{i_A \cdot k_x} - \Lambda_{.i_A})_{.i_x j_x} \end{aligned} \tag{2.16}$$

Because we are considering broken world lines, it could be the case that we require a general solution to equation (2.16) in order to satisfy the boundary conditions at the end points of the world lines. We therefore write the general solution to (2.16) as

$$\bar{G}^{k_x}{}_{i_A \cdot k_x} - \Lambda_{.i_A} = -\bar{G}_{.i_A} + \psi_{i_A} \tag{2.17}$$

here ψ_{i_A} is any four vector satisfying

$$g^{i_x j_x} \psi_{i_A \cdot i_x j_x} = 0 \tag{2.18}$$

In order for our theory to be self-consistent we must, of course, check that the vector potential, defined by (2.11), satisfies the assumed arbitrary gauge given by (2.12). Taking the divergence of (2.13) and using equation (2.17) we have

$$4\pi e_a \int A_{.i_A} da^{i_A} = A^{(a)\,i_x}{}_{.i_x}$$
$$= 4\pi e_a \int \bar{G}^{i_x}{}_{i_A.i_x} da^{i_A}$$
$$= -4\pi e_a \int \bar{G}_{.i_A} da^{i_A}$$
$$+ 4\pi e_a \int A_{.i_A} da^{i_A} + 4\pi e_a \int \psi_{i_A} da^{i_A} \qquad (2.19)$$

From equation (2.2) we see that the condition for consistency therefore becomes

$$e_a f C^{(a)}(X) = e_a \int \bar{G}_{.i_A} da^{i_A} = e_a \int \psi_{i_A} da^{i_A} \qquad (2.20)$$

since the assumed gauge cancels on both sides of equation (2.19). Comparison of the equations (2.3) and (2.18), which define the functions $\bar{G}_{.i_A}$ and ψ_{i_A}, clearly shows that equation (2.20) can never hold unless the creation field itself vanishes. Our theory is therefore self-inconsistent.

At this stage it is worth drawing attention to the fact that when summing all our equations to obtain the total contributions to the fields and setting $\psi_{i_A} = 0$, it is possible to have

$$\sum_a e_a C^{(a)}(X) = 0 \qquad (2.21)$$

which will give a consistent result in (2.19). As pointed out by Hoyle & Narlikar (1964b), this could be the case if world lines are correlated in such a way that charge is always conserved. If we could restrict our considerations to the total fields arising, then the condition (2.21) would produce a consistent theory. However, this is virtually impossible since equation (2.21) itself requires the introduction of a partial creation field. Our general conclusion must therefore be that the action defined by Hoyle and Narlikar and given by equations (2.9) and (2.10) does not correspond to the action of the usual field theory given in equation (2.5).

3. *The New Action and the Modified Maxwellian Equations*

The question now arises as to what form Maxwell's equations must take in order to be consistent with the actions given in (2.9) and (2.10). Consider the modified Maxwellian equations

$$F^{(a)\,i_x j_x}{}_{.j_x} = 4\pi j^{(a)\,i_x} - 4\pi e_a f\, C^{(a)\,i_x}_{\;\cdot} \qquad (3.1)$$

Introducing the vector potential, equation (3.1) becomes

$$g^{j_x k_x} A^{(a) i_x}{}_{.j_x k_x} - A^{(a) k_x}{}_{.k_x}{}^{i_x} + R^{i_x}{}_{k_x} A^{(a) k_x} = -4\pi j^{(a) i_x} + 4\pi e_a f\, C^{(a)}{}_{.}{}^{i_x} \tag{3.2}$$

Choosing the gauge condition such that

$$A^{(a) k_x}{}_{.k_x} = -4\pi e_a f\, C^{(a)}(X) \tag{3.3}$$

and defining the vector potential by the relation

$$A^{(a)}{}_{k_x} = 4\pi e_a \int \bar{G}_{k_x i_A}\, da^{i_A} \tag{3.4}$$

we see, from (3.2), that the vector Green's function satisfies the equation

$$g^{j_x k_x} \bar{G}_{i_x i_A . j_x k_x} + R_{i_x}{}^{k_x} \bar{G}_{k_x i_A} = -\bar{g}^{-1/2} \bar{g}_{i_x i_A} \delta^{(4)}(X, A) \tag{3.5}$$

Using the identity of equation (2.15) we see from (2.3) and (3.5) that the relation between the vector and scalar Green's functions takes the form

$$\bar{G}^{i_x}{}_{i_A . i_x} = -\bar{G}_{.i_A} \tag{3.6}$$

For consistency we once again require that the vector potential defined by (3.4) has a gauge given by (3.3). Taking the divergence of (3.4) and using (2.2) and (3.6), we have

$$\begin{aligned} -4\pi e_a f\, C^{(a)} &= A^{(a) k_x}{}_{.k_x} \\ &= 4\pi e_a \int \bar{G}^{k_x}{}_{i_A . k_x}\, da^{i_A} \\ &= -4\pi e_a \int \bar{G}_{.i_A}\, da^{i_A} \end{aligned} \tag{3.7}$$

From the definition of the creation field, equation (2.2), we see that our consistency condition is therefore satisfied and leads us to conclude that the Hoyle and Narlikar action (2.10) corresponds to the modified Maxwellian equations (3.1).

From (2.2), (2.8), (3.5) and (3.6) it also follows that the source equation is unchanged from that of the usual field theory and takes the form

$$f\, C^{(a)}{}_{.}{}^{i_x}{}_{i_x} = j^{(a) i_x}{}_{.i_x} \tag{3.8}$$

The question now arises as to what the corresponding modification of the action of the usual field theory must be, in order to produce equations (3.1) and (3.8) and therefore be equivalent to the action (2.10). First, from (3.1), we see that a C-field-vector potential interaction term must be added to (2.5) in order to give the C-field contribution to Maxwell's equations. Secondly, since this additional term taken alone will modify the source equation, we must add another term depending on the C-field only which will balance this contribution. The action corresponding to (2.10) must therefore be

$$
\begin{aligned}
L = & \frac{1}{16\pi G}\int R\sqrt{(-g)}\,d^4x - \sum_a m_a \int da \\
& - \frac{1}{16\pi}\int F_{lm}F^{lm}\sqrt{(-g)}\,d^4x - \sum_a e_a \int A_i\,da^i \\
& - \tfrac{1}{2}f\int C_{.m}C_{.}^{m}\sqrt{(-g)}\,d^4x + \sum_a \int C_{.i}\,da^i \\
& - \sum_a e_a f\int C^{(a)}{}_{.i}A^i\sqrt{(-g)}\,d^4x + 2\pi f^2\sum_a\sum_b e_a e_b\int C^{(a)}C^{(b)}\sqrt{(-g)}\,d^4x
\end{aligned}
\tag{3.9}
$$

An independent variation of the A_i's gives, from the third, fourth and seventh terms, Maxwell's equations in the form

$$F^{ij}{}_{.j} = 4\pi j^i - 4\pi\sum_a e_a C^{(a)}{}_{.}{}^{i} \tag{3.10}$$

Variations of C, keeping the metric and A_i constant, gives from the fifth and sixth terms the source equation (3.8). The final two terms make no contribution to this since they cancel in the following manner:

$$
\begin{aligned}
& \int\left\{+\sum_a e_a f A^i{}_{.i} + 2\pi f^2\sum_a\sum_b e_a e_b 2C^{(b)}\right\}\delta C^{(a)}\sqrt{(-g)}\,d^4x \\
& = +\int\sum_a e_a f\left\{A^i{}_{.i} + \sum_b 4\pi e_b C^{(b)}\right\}\delta C^{(a)}\sqrt{(-g)}\,d^4x \\
& = 0
\end{aligned}
$$

in virtue of the relationship of equation (3.3).

The only thing we have not yet considered is the contribution of the final two terms to the energy-momentum tensor. Varying the metric we find that the third and fifth terms contribute an amount

$$\frac{1}{4\pi}\{\tfrac{1}{4}g^{ik}F^{lm}F_{lm} - F^{il}F^{k}{}_{l}\} - f\{C_{.}^{i}C_{.}^{k} - \tfrac{1}{2}g^{ik}C_{.m}C_{.}^{m}\} \tag{3.11}$$

whilst the final two terms contribute

$$2f\sum_a e_a(C^{(a)}{}_{.}{}^{i}A^k - \tfrac{1}{2}g^{ik}C^{(a)}{}_{.m}A^m) + 2\pi f^2 g^{ik}\sum_a\sum_b e_a e_b C^{(a)}C^{(b)} \tag{3.12}$$

Once again taking note of the relationship

$$A^i{}_{.i} = -4\pi f\sum_a e_a C^{(a)} \tag{3.13}$$

we see, from (3.11) and (3.12), that the total contribution to the energy-momentum tensor can be written as

$$\underset{\text{em}}{T^{ik}} + \underset{C}{T^{ik}} = (4\pi)^{-1}\{\tfrac{1}{4}F^{lm}F_{lm}g^{ik} - F^{il}F^{k}{}_{l}\} - f\{C_{.}^{i}C_{.}^{k} - \tfrac{1}{2}g^{ik}C_{.}^{l}C_{.l}\}$$
$$+ (4\pi)^{-1}\{-2A^{i}A^{l}{}_{.l}{}^{k} + g^{ik}A^{m}{}_{.ml}A^{l} + \tfrac{1}{2}A^{m}{}_{.m}A^{l}{}_{.l}g^{ik}\} \qquad (3.14)$$

How does this equation compare with the result obtained by Hoyle and Narlikar when varying the action (2.10)? They obtained the following form for the energy-momentum tensor

$$\underset{\text{em}}{T^{ik}} + \underset{C}{T^{ik}} = (4\pi)^{-1}\sum_{a<b}\sum\{\tfrac{1}{2}g^{ik}F^{(a)lm}F^{(b)}{}_{lm} - F^{(a)il}F^{(b)k}{}_{l} - F^{(b)il}F^{(a)k}{}_{l}\}$$
$$- f\sum_{a<b}\sum\{C^{(a)}{}_{.}^{i}C^{(b)}{}_{.}^{k} + C^{(b)}{}_{.}^{i}C^{(a)}{}_{.}^{k} - g^{ik}C^{(a)}{}_{.}^{l}C^{(b)}{}_{.l}\}$$
$$- (2\pi)^{-1}\sum_{a<b}\sum\{g^{mk}(A^{(a)i}A^{(b)l}{}_{.lm} + A^{(b)i}A^{(a)l}{}_{.lm})\}$$
$$+ (4\pi)^{-1}\sum_{a<b}\sum\{g^{ik}(A^{(a)m}A^{(b)l}{}_{.lm} + A^{(b)m}A^{(a)l}{}_{.lm}$$
$$+ A^{(a)l}{}_{.l}A^{(b)m}{}_{.m})\} \qquad (3.15)$$

Clearly (3.15) reduces to (3.14) in the smooth fluid approximation. This is the final point required for the two theories to be in agreement. We may therefore sum up as follows. The action functionals (2.9) and (2.10) postulated by Hoyle and Narlikar are equivalent, not to the action (2.5) of the usual field theory, but the modified form given by (3.9). These theories then give Maxwell's equations in the modified form (3.1), but the source equation still retains its usual form (3.8). The extra two terms in (3.9) culminate in an additional contribution to the energy-momentum tensor by the amount given in (3.12). In general the modified form of Maxwell's equations, given by (3.10), no longer possesses the usual conformal invariance property. In order to retain this property it is necessary to postulate the conservation of charge in the manner of (2.21). This loss of conformal invariance, of course, invalidates the use of the flat space vector potential in order to determine the field strengths in conformally flat spaces. This could have repercussions, for example in the work of Hogarth (1962) and Hoyle & Narlikar (1964a), in determining the consistency conditions for various cosmologies. Finally we note that, since the additional term in Maxwell's equations (3.10) is the derivative of a scalar function, it can always be absorbed into a change of gauge.

4. *The Action for Direct Particle Fields*

If we differentiate the scalar wave equation (2.3), first with respect to the field point, then with respect to the particle position, and use the commutation properties satisfied by the Ricci tensor, then we can write (2.3) in the form

$$g^{j_x k_x}\bar{G}(X,A)_{.i_x i_A j_x k_x} + R_{i_x}{}^{k_x}\bar{G}(X,A)_{.k_x i_A} = -(g^{-1/2}\delta^{(4)})_{.i_x i_A} \qquad (4.1)$$

Comparison of (3.5) and (4.1) shows that the wave equations satisfied by $\bar{G}_{i_x i_A}$ and $\bar{G}_{.i_x i_A}$ are identical except for the source terms on the right. We can therefore introduce a new Green's function $\tilde{G}_{i_x i_A}$ defined by

$$\tilde{G}_{i_x i_A} = \bar{G}_{i_x i_A} - \lambda \bar{G}_{.i_x i_A} \tag{4.2}$$

where λ is an arbitrary constant. The new Green's function will satisfy the equation

$$g^{j_x k_x} \tilde{G}_{i_x i_A \cdot j_x k_x} + R_{i_x}{}^{k_x} \tilde{G}_{k_x i_A} = -\bar{g}^{-1/2} \bar{g}_{i_x i_A} \delta^{(4)} + \lambda (\bar{g}^{-1/2} \delta^{(4)})_{.i_x i_A} \tag{4.3}$$

Our next step is to define a new vector potential for the electromagnetic field by the relation

$$A^{(a)}{}_{i_x} = 4\pi e_a \int \tilde{G}_{i_x i_A} da^{i_A} \tag{4.4}$$

Substituting (4.4) into (4.3) and using the definition (2.2), yields Maxwell's equations in the form of (3.2). Noting that the gauge associated with (4.4) is, by (3.6),

$$A^{(a) i_x}{}_{.i_x} = 4\pi e_a \left\{ - \int \bar{G}_{.i_A} da^{i_A} + \lambda \int (\bar{g}^{-1/2} \delta^{(4)})_{.i_A} da^{i_A} \right\} \tag{4.5}$$

and taking the divergence of (4.3), we obtain the source equation (3.8). The new vector potential (4.4) therefore satisfies both the modified Maxwellian equations and the source equation identically. The parameter λ in (4.2) corresponds to the arbitrariness of gauge associated with Maxwell's equations.

There is now no need to consider the C-field in our equations since its presence is implicit in the new vector potential (4.4). Finally, in view of (4.4) the direct particle action (2.10) can be written in the simpler form

$$A = \frac{1}{16\pi G} \int R\sqrt{(-g)}\, d^4 x - \sum_a m_a \int da$$
$$- \sum_{a<b} \sum 4\pi e_a e_b \iint \tilde{G}_{i_A i_B} da^{i_A} db^{i_B} \tag{4.6}$$

where the parameter λ takes the value

$$\lambda = (4\pi e_a e_b f)^{-1} \tag{4.7}$$

Acknowledgements

I wish to thank Dr. C. Gilbert for valuable discussions.

References

Hogarth, J. E. (1962). *Proceedings of the Royal Society*, *A*, **267**, 365.

Hoyle, F. and Narlikar, J. V. (1964a). *Proceedings of the Royal Society*, *A*, **277**, 1; (1964b). **282**, 178; (1964c). **282**, 184.